A DECOMPOSITION THEOREM FOR BINARY MATROIDS WITH NO PRISM MINOR

S. R. Kingan [1]
Department of Mathematics
Brooklyn College, City University of New York
Brooklyn, NY 11210
skingan@brooklyn.cuny.edu
Manoel Lemos [2]
Departamento de Matematica
Universidade Federal de Pernambuco
Recife, Pernambuco, 50740-540, Brazil
manoel@dmat.ufpe.br

Abstract. The prism graph is the dual of the complete graph on five vertices with an edge deleted, $K_5\backslash e$. In this paper we determine the class of binary matroids with no prism minor. The motivation for this problem is the 1963 result by Dirac where he identified the simple 3-connected graphs with no minor isomorphic to the prism graph. We prove that besides Dirac's infinite families of graphs and four infinite families of non-regular matroids determined by Oxley, there are only three possibilities for a matroid in this class: it is isomorphic to the dual of the generalized parallel connection of F_7 with itself across a triangle with an element of the triangle deleted; it's rank is bounded by 5; or it admits a non-minimal exact 3-separation induced by the 3-separation in P_9. Since the prism graph has rank 5, the class has to contain the binary projective geometries of rank 3 and 4, F_7 and $PG(3,2)$, respectively. We show that there is just one rank 5 extremal matroid in the class. It has 17 elements and is an extension of R_{10}, the unique splitter for regular matroids. As a corollary, we obtain Dillon, Mayhew, and Royle's result identifying the binary internally 4-connected matroids with no prism minor [5].

1. Introduction

In a decomposition result, a more complicated matroid is broken down into simpler components. The fact that such simplifications exist is surprising and indicative of deep order in the structure of infinite classes of matroids. In 1980 Seymour decomposed the class of regular matroids, begining a flourishing genre of such structural results [9]. A matroid is *regular* if it has no minor isomorphic to the Fano matroid F_7 or its dual F_7^*. To decompose regular matroids, he developed the Splitter Theorem, a Decomposition Theorem, and the notion of 3-sums. The Splitter Theorem describes how 3-connected matroids can be systematically built-up and the Decomposition Theorem describes the conditions under which a specific type of separation in a matroid gets carried forward to all matroids containing it. The proof of the decomposition of regular matroids consists of three main parts. The first part establishes that a 3-connected regular matroid is graphic or cographic or has a minor isomorphic to R_{10} or R_{12}. The matroid R_{10} is a splitter for regular matroids. This means no 3-connected regular matroid contains it (other than R_{10} itself). So the building-up process stops

[1]The first author is partially supported by PSC-CUNY grant number 64181-00 42
[2]The second author is partially supported by CNPq under grant number 300242/2008-05.

at R_{10}. The second part establishes that R_{12} has a non-minimal exact 3-separation that carries forward in all matroids containing it. The third part establishes that 3-connected regular matroids can be pieced together from graphic and co-graphic matroids using the operation of 3-sums. It is well-known that matroids that are not 3-connected can be pieced together from 3-connected matroids using the operations of 1-sum and 2-sum, so it sufficies to focus on the 3-connected members of a class.

We present the decomposition of binary matroids with no minor isomorphic to the prism graph. To decompose this class we used a strengthening of the Splitter Theorem [3] and a decomposition theorem by Mayhew, Royle, and Whittle [4]. The class of binary matroids with no prism minor is quite different from the class of regular matroids, but also similar in the sense that there are several special matroids in it and one of them has a separation that carries forward. The role of R_{12} is played by the non-regular matroid P_9. The prism graph, shown in Figure 1, is the dual of the complete graph on five vertices, K_5 with one edge deleted. It is denoted as $(K_5\backslash e)^*$.

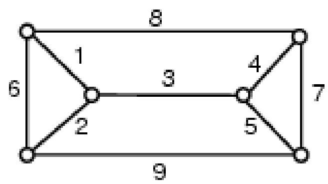

FIGURE 1. The prism graph

A matrix representation for the prism graph is shown below.

$$\left[\begin{array}{c|cccc} & 1 & 0 & 1 & 0 \\ & 1 & 0 & 0 & 1 \\ I_5 & 0 & 0 & 1 & 1 \\ & 0 & 1 & 1 & 0 \\ & 0 & 1 & 0 & 1 \end{array}\right]$$

The origin of this excluded minor problem can be traced to 1963 when Dirac determined the extremal graphs without two vertex disjoint cycles [1]. Excluding two vertex-disjoint cycles in a 3-connected graph is equivalent to excluding $(K_5\backslash e)^*$ as a minor. For $r \geq 3$, let W_r denote the wheel with r spokes, and for $p \geq 3$, let $K_{3,p}$ denote the complete bipartite graph with three vertices in one class and p vertices in the other class. Let $K'_{3,p}$, $K''_{3,p}$, and $K'''_{3,p}$ denote the graphs obtained from $K_{3,p}$ by adding one, two, and three edges, respectively, joining vertices in the class containing three vertices. Dirac proved that a simple 3-connected graph has no minor isomorphic to $(K_5\backslash e)^*$ if and only if it is isomorphic to W_r for some $r \geq 3$, K_5, $K_5\backslash e$, $K_{3,p}$, $K'_{3,p}$, $K''_{3,p}$ or $K'''_{3,p}$ for some $p \geq 3$. In 1984 Robertson and Seymour published a note where they proved that a simple 3-connected graph with at least four vertices has no minor isomorphic to $K_5\backslash e$ if and only if it is isomorphic to $(K_5\backslash e)^*$, $K_{3,3}$, or W_r for some $r \geq 3$ [8]. In 1996 Kingan characterized the 3-connected regular matroids with no minor isomorphic to $M^*(K_5\backslash e)$ [2, 2.1].

Theorem 1.1. *M is a 3-connected regular matroid with no minor isomorphic to $M^*(K_5\backslash e)$ if and only if M is isomorphic to $M(W_r)$ for some $r \geq 3$, $M(K_5)$, $M(K_5\backslash e)$, $M^*(K_{3,3})$, $M(K_{3,p})$, $M(K'_{3,p})$, $M(K''_{3,p})$, or $M(K'''_{3,p})$ for some $p \geq 3$, or R_{10}.*

Some matroids like R_{10} play a significant role in structural results. This class contains one such significant matroid called E_5. It is a self-dual non-regular internally 4-connected single-element

extension of $M(K_{3,3})$. It is also the splitter for the class of binary matroids with no minor isomorphic to the prism graph, it dual, or the binary affine cube $AG(3,2)$ [2]. A matrix representation is shown below.

$$E_5 = \left[\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 0 \\ I_5 & 1 & 1 & 0 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

In order to characterize the class of binary non-regular 3-connected matroids with no prism minor, we flag a particular binary non-regular 9-element rank-4 matroid known as P_9 and prove that besides a few exceptional matroids, all the matroids in the class have P_9 as a 3-decomposer. This means the matroid has a non-minimal exact 3-separation induced by the non-minimal exact 3-separation of P_9. P_9 is the generalized parallel connection, $P_\triangle(F_7, W_3)$, of F_7 and W_3 across a triangle, with the rim element of the triangle deleted. Note that $P_9 \cong H_4$, mentioned above. A matrix representation for P_9 is given below.

$$P_9 = \left[\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 \end{array}\right]$$

The matroid P_9 appears in [5] where Oxley characterized the 3-connected binary non-regular matroid with no minors isomorphic to P_9 or P_9^*. Members of this class are the infinite families Z_r, Z_r^*, $Z_r \backslash b_r$ and $Z_r \backslash c_r$. The matroid Z_r is a $(2r+1)$-element rank-r non-regular matroid. It can be represented by the binary matrix $[I_r|D]$ where D has $r+1$ columns labeled $b_1, \ldots, b_r, c_r$. The first r columns in D have zeros along the diagonal and ones elsewhere. The last column is all ones. Note that $Z_4 \backslash c_4 \cong AG(3,2)$ and $Z_4 \backslash b_4 \cong S_8$, where S_8 and $AG(3,2)$ are the two non-isomorphic single-element extensions of F_7^*. The next theorem appears in [6].

Theorem 1.2. *M is a 3-connected binary non-regular matroid with no minor isomorphic to P_9 or P_9^* if and only if M is isomorphic to F_7, F_7^*, Z_r, Z_r^*, $Z_r \backslash b_r$, or $Z_r \backslash c_r$, for some $r \geq 4$.* □

It is easy to show that the above infnite families do not have a prism minor. As a consequence we may conclude a binary non-regular matroid with no prism minor is either one of the infinite families mentioned in Theorems 1.2 or it has a P_9-minor. Like R_{12}, P_9 has a non-minimal exact 3-separation in it. However, unlike R_{12}, the separation in P_9 does not extend to all matroids containing it. Nonetheless, we are able to identify all of the exceptions. Clearly all binary non-regular 3-connected rank 4 matroids have no prism minor, since the prism graph has rank 5. Thus $PG(3,2)$ and all of its deletion minors have no prism minor. We prove that besides one 11-element rank-6 matroid, all the exceptions have rank at most 5. The rank 6 exception is the dual of the generalized parallel connection of F_7 with itself across a triangle, with an element of the triangle deleted, denoted as $(P_\triangle(F_7, F_7) \backslash e)^*$. We are now ready to state the main results of this paper.

Theorem 1.3. *Suppose M is a 3-connected binary non-regular matroid with no $M^*(K_5 \backslash e)$-minor. Then one of the following holds:*

(i) *M is isomorphic to Z_r, Z_r^*, $Z_r \backslash b_r$, or $Z_r \backslash c_r$, for some $r \geq 4$;*
(ii) *P_9 is a 3-decomposer for M;*
(iii) *M is isomorphic to $(P_\triangle(F_7, F_7) \backslash z)^*$; or*
(iv) *M has rank at most 5.*

A detailed analysis of rank 5 binary matroids reveals that all of them are restriction minors of one particular 17-element matroid R_{17}, that is an extension of E_5 and R_{10} [2]. A matrix representation for R_{17} is shown below.

$$R_{17} = \left[\begin{array}{c|cccccccccccc} & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ I_5 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$

As a corollary of Theorem 1.3, we obtain the following characterization of binary matroids with no prism minor.

Theorem 1.4. *Suppose M is a 3-connected binary matroid with no $M^*(K_5\backslash e)$-minor. Then either P_9 is a 3-decomposer for M or M^* is isomorphic to one of the following matroids:*

(i) *$M(W_r)$ for some $r \geq 3$, $M(K_{3,p})$, $M(K'_{3,p})$, $M(K''_{3,p})$ or $M(K'''_{3,p})$, for some $p \geq 3$; or*
(ii) *Z_r, Z_r^*, $Z_r\backslash b_r$, or $Z_r\backslash c_r$, for some $r \geq 4$;*
(iii) *F_7, $(P_\triangle(F_7, F_7)\backslash z)^*$; or*
(iv) *$PG(3,2)$ or R_{17} or one of their 3-connected restrictions.*

Note that, $Z_4\backslash c_4 \cong AG(3,2)$, $Z_4\backslash b_4 \cong S_8$, $M(K_5)$, $M(K_5\backslash e)$, $M^*(K_{3,3})$, F_7^* and P_9 are restrictions (deletion-minors) of $PG(3,2)$. The matroid $(P_\triangle(F_7, F_7)\backslash z)^*$ has rank 6 and 10 elements. Therefore, $P_\triangle(F_7, F_7)\backslash z$ is a rank 4, 10-element matroid and a restriction of $PG(3,2)$. The matroid R_{10}, P_9^*, and E_5 are restrictions of R_{17}. So we do not have to list these matroids explicitly.

In the next section we give the statement and proof of the decomposition theorem that forms a key component of Theorem 1.3. In Section 4 we prove Theorem 1.4. We also determine the class of binary matroids with no $M(K_5\backslash e)$-minor and the class with neither $M(K_5\backslash e)$ nor $M^*(K_5\backslash e)$-minor.

2. **Proof of Theorem 1.3**

The matroid terminology follows Oxley [7]. We should note that the matroid corresponding to the matrix labeled A is called $M[A]$ and not just A. However, we refer to large numbers of matrices in this paper and with the reader's understanding treat the matrix and matroid as synonymous.

Let M be a matroid and X be a subset of the ground set E. The *connectivity function* λ is defined as $\lambda(X) = r(X) + r(E - X) - r(M)$. Observe that $\lambda(X) = \lambda(E - X)$. For $k \geq 1$, a partition (A, B) of E is called a k-separation if $|A| \geq k$, $|B| \geq k$, and $\lambda(A) \leq k - 1$. When $\lambda(A) = k - 1$, we call (A, B) an *exact k-separation.* When $\lambda(A) = k - 1$ and $|A| = k$ or $|B| = k$ we call (A, B) a *minimal exact k-separation.* For $n \geq 2$, we say M is *n-connected* if M has no k-separation for $k \leq n - 1$. A matroid is *internally n-connected* if it is n-connected and has no non-minimal exact n-separations. In particular, a simple matroid is 3-connected if $\lambda(A) \geq 2$ for all partitions (A, B) with $|A| \geq 3$ and $|B| \geq 3$. A 3-connected matroid is *internally 4-connected* if $\lambda(A) \geq 3$ for all partitions (A, B) with $|A| \geq 4$ and $|B| \geq 4$. For example, E_5 is internally 4-connected, but P_9 is not. In the matrix representation of P_9 in Section 1, it has a non-minimal exact 3-separation (A, B) where $A = \{1, 2, 5, 6\}$.

Let $\mathcal{M}$ be a class of matroids closed under minors and isomorphisms. Let $k \geq 1$ and N be a matroid belonging to $\mathcal{M}$ having an exact k-separation (A, B). Let $M \in \mathcal{M}$ having an N-minor.

We say that N is a *k-decomposer* for M having (A, B) as an *inducer* provided M has a k-separation (X, Y) such that $A \subseteq X$ and $B \subseteq Y$.

It is well known that every non-regular binary matroid has a minor isomorphic to F_7 or F_7^*. Thus we may consider this our starting point for any investigation of non-regular matroids. As mentioned earlier, $AG(3, 2)$ and S_8 are the two non-isomorphic 3-connected single-element extensions of F_7^*. The matroid S_8 has two non-isomorphic 3-connected single-element extensions P_9 and Z_4. The matroid $AG(3, 2)$ has one 3-connected single-element extension, Z_4. As noted earlier, P_9 has a non-minimal exact 3-separation (and cosequently so does P_9^*).

We begin by proving that P_9 or P_9^* are 3-decomposers for a certain class of matroids. To do so we use the following result by Mayhew, Royle, and Whittle in [3, 2.10]. Then, we will prove the stronger statement that P_9^* is not relevant and, in fact, P_9 is the required 3-decomposer (with one exception). We end by showing that the rank of the exceptional matroids that do not have P_9 as a 3-decomposer is bounded by 5. This portion requires the Strong Splitter Theorem [3, 1.4].

Lemma 2.1. *Suppose $\mathcal{M}$ is a class of matroids closed under minors and isomorphism and let $N \in \mathcal{M}$ be a 3-connected matroid with $|E(N)| \geq 8$ and a non-minimal exact 3-separation (A, B) where A is a 4-element circuit and a cocircuit. If A is a circuit and cocircuit in every 3-connected single-lement extension and coextension of N in $\mathcal{M}$, then N is a 3-decomposer for every matroid in $\mathcal{M}$ with an N-minor.* $\square$

The significance of the above decomposition result is that it makes it easy to determine whether or not a non-minimal exact 3-separation carries forward. Compare the criteria in this result to the original criteria in Seymour's Decomposition Theorem [8, 9.1].

Lemma 2.2. *Suppose N is a 3-connected proper minor of a 3-connected matroid M such that, if N is a wheel or whirl then M has no larger wheel or whirl-minor, respectively. Further, suppose $m = r(M) - r(N)$. Then there is a sequence of 3-connected matroids $M_0, M_1, \ldots, M_n$, for some integer $n \geq m$, such that*

(i) *$M_0 \cong N$;*
(ii) *$M_n = M$;*
(iii) *for $k \in \{1, 2, \ldots, m\}$, $r(M_k) - r(M_{k-1}) = 1$ and $|E(M_k) - E(M_{k-1})| \leq 3$; and*
(iv) *for $m < k \leq n$, $r(M_k) = r(M)$ and $|E(M_k) - E(M_{k-1})| = 1$.*

Moreover, when $|E(M_k) - E(M_{k-1})| = 3$, for some $1 \leq k \leq m$, $E(M_k) - E(M_{k-1})$ is a triad of M_k.

The significance of the Strong Splitter Theorem is that we can obtain, up to isomorphism, M starting with N and at each step doing a 3-connected single-element extension or coextension, such that at most two consecutive single-element extensions occur in the sequence (unless the rank of the matroids involved are r). Moreover, if two consecutive single-element extensions by elements $\{e, f\}$ are followed by a coextension by element g, then $\{e, f, g\}$ form a triad in the resulting matroid. This greatly reduces the computations we need to establish a bound on the rank.

Proof of Theorem 1.3. Since M is non-regular, M has a minor isomorphic to F_7 or F_7^*. Since F_7 has no binary extensions, we may assume M has a minor isomorphic to F_7^*. The 8-element binary simple extensions of F_7^* are $AG(3, 2)$ and S_8 and the 9-element simple extensions are P_9 and Z_4. We present the proof as a series of claims.

Claim 1. If M has no P_9 nor P_9^*-minor, then M is isomorphic to F_7, F_7^*, Z_r, Z_r^*, $Z_r \backslash c_r$, or $Z_r \backslash b_r$ for $r \geq 4$.

Proof. Theorem 1.2 identifies the above families as the binary non-regular 3-connected matroids with no P_9 nor P_9^*-minor. To prove Theorem 1.2, Oxley proves that for $r \geq 4$, Z_r, Z_r^*, $Z_r \backslash c_r$, and $Z_r \backslash b_r$ have no $M(W_4)$-minor [5, Theorem 2.1]. Since $M^*(K_5 \backslash e)$ and $M(K_5 \backslash e)$-minor have an $M(W_4)$-minor, we may conclude that Z_r, Z_r^*, $Z_r \backslash c_r$, and $Z_r \backslash b_r$ have no minor isomorphic to $M^*(K_5 \backslash e)$ nor $M(K_5 \backslash e)$.

Returning to the proof of the theorem, Claim 1 implies that M has a minor isomorphic to P_9 or P_9^*. Now, P_9 has three simple non-isomorphic binary single-element extensions shown below. Adding column [1110] gives D_1, adding columns [1001], [0101], [0110], or [1010] gives D_2, and adding column [0011] gives D_3. This is concisely summarized in Table 1a and representative matrices for D_1, D_2, and D_3 are given below. Note that, Table 1a gives the extensions of P_9. Columns in bold are the ones used to form the matrices. The final three rank 4 matrices are $PG(3,2)$, $PG(3,2) \backslash e$ and $PG(3,2) \backslash \{e, f\}$.)

$$D_1 = \left[\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right] D_2 = \left[\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 1 & 0 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 1 \end{array}\right] D_3 = \left[\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 0 \\ I_4 & 1 & 0 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 1 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 1 \end{array}\right]$$

Matroid	**Extension Columns**	**Name**
P_9	[**1110**]	D_1
	[**1001**] [0101] [0110], [1010]	D_2
	[**0011**]	D_3
D_1	[0101] [0110] [1001] [1010]	X_1
	[**0011**]	X_2
D_2	[**1010**] [1110]	X_1
	[**0011**] [0101] [0110]	X_3
D_3	[1110]	X_2
	[0101] [0110] [1001] [1010]	X_3
X_1	[**0011**] [0101] [0110]	Y_1
	[**1110**]	Y_2
X_2	[0101] [0110] [1001] [1010]	Y_1
X_3	[0101] [0110] [1010] [1110]	Y_1

Table 1a: Rank 4 extensions of P_9

P_9 has eight cosimple non-isomorphic single-element coextensions (see Table 1b). When coextending a rank-4 matrix the column $[0,0,0,0,1]$ is added as the fifth element and a new row is added at the bottom of the right hand side of the matrix. The coextended element is column 5.

$$E_1 = \left[\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 0 & 0 \end{array}\right] E_2 = \left[\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 1 & 1 \end{array}\right] E_3 = \left[\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 0 & 1 \end{array}\right] E_4 = \left[\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 \\ & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$E_5 = \left[\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 \\ & 1 & 0 & 1 & 0 & 0 \end{array}\right] E_6 = \left[\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 \\ & 0 & 0 & 1 & 0 & 1 \end{array}\right] E_6^* = \left[\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 \\ & 0 & 0 & 1 & 1 & 1 \end{array}\right] E_7 = \left[\begin{array}{c|ccccc} & 0 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 \\ & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Coextension Rows	**Name**
[11000] [11111]	E_1
[11011] [11100]	E_2
[11001] [11101]	E_3
[01001] [01010] [01101] [01110] [10001] [10010] [10101] [10110]	E_4
[01011] [01100] [10011] [10100]	E_5
[00101] [00110]	E_6
[00111]	E_6^*
[00011]	E_7

Table 1b: Single-element coextensions of P_9

Claim 2. If M has a P_9-minor, but no D_2, D_2^*, E_4, or E_5-minor, then P_9 or P_9^* is a 3-decomposer for M.

Proof. As mentioned earlier, P_9 has a non-minimal exact 3-separation (A, B) where $A = \{1, 2, 5, 6\}$ is both a circuit and a cocircuit. It is easy to check that the set $A = \{1, 2, 5, 6\}$ is both a circuit and a cocircuit in D_1 and D_3, whereas D_2 is internally 4-connected. Next, the set $A = \{1, 2, 5, 6\}$ corresponds to $A' = \{1, 2, 6, 7\}$ in the coextension since the fifth column is the coextended element. It can be checked that $\{1, 2, 6, 7\}$ is both a circuit and a cocircuit in E_1, E_2, E_3, E_6, E_6^*, and E_7. Since E_4 and E_5 are self-dual, the claim follows from Lemma 2.1.

Claim 3. If M has no D_2, D_2^*, E_4 or E_5-minor, then one of the following hold:

(i) M is isomorphic to F_7, F_7^*, Z_r, Z_r^*, $Z_r \backslash b_r$, or $Z_r \backslash c_r$, for some $r \geq 4$; or
(ii) P_9 or P_9^* is a 3-decomposer for M

Proof. If M has no P_9 or P_9^*-minor, then (i) follows from Claim 1. So, we may assume that M has a P_9 or P_9^* as a minor. If M has a P_9-minor, then P_9 is a 3-decomposer for M. Otherwise, M has a P_9^*-minor and by duality P_9^* is a 3-decomposer for M.

Claim 4. If M has an E_5-minor, then either $M \cong E_5$ or M has a D_2 or D_2^*-minor.

Proof. The matroid E_5 is self-dual and has seven non-isomorphic binary 3-connected single-element extensions, shown in Table 2a. All of them have a minor isomorphic to D_2 and the claim follows.

Extension Columns	**Name**	**Contraction-minor**	**Deletion-minor**
[00101] [00110] [01011] [01100]	A	D_2	E_5 E_6^*, E_7, $K'_{3,3}$
[10011]	B	D_2	E_5, $K'_{3,3}$, R_{10}
[11001] [11101]	C	D_2	E_5, E_6^*, E_7, E_3
[00011] [00111] [01001] [01101]		D_2	E_4
01010] [01110]		D_2	E_4
[10001] [10010] [11011] [11100]		D_2	E_4
[10101] [10110] [11000] [11111]		D_2	E_4

Table 2a: Single-element extensions of E_5

Returning to the proof of the theorem, we will determine which of the extensions and coextensions of P_9 have minors isomorphic to $M^*(K_5 \backslash e)$ or $M(K_5 \backslash e)$. A matrix representation for the graph $K_5 \backslash e$ is given below.

$$K_5 \backslash e = \left[\begin{array}{c|ccccc} & 1 & 0 & 0 & 1 & 1 \\ I_4 & 1 & 1 & 0 & 0 & 0 \\ & 0 & 1 & 1 & 0 & 1 \\ & 0 & 0 & 1 & 1 & 0 \end{array} \right]$$

It has three binary non-isomorphic single-element extensions, K_5, D_2, and D_3 (see Table 3a).

$$K_5 = \left[\begin{array}{c|cccccc} & 1 & 0 & 0 & 1 & 1 & 0 \\ I_4 & 1 & 1 & 0 & 0 & 0 & 1 \\ & 0 & 1 & 1 & 0 & 1 & 0 \\ & 0 & 0 & 1 & 1 & 0 & 1 \end{array} \right] D_2 = \left[\begin{array}{c|cccccc} & 1 & 0 & 0 & 1 & 1 & 0 \\ I_4 & 1 & 1 & 0 & 0 & 0 & 1 \\ & 0 & 1 & 1 & 0 & 1 & 1 \\ & 0 & 0 & 1 & 1 & 0 & 1 \end{array} \right] D_3 = \left[\begin{array}{c|cccccc} & 1 & 0 & 0 & 1 & 1 & 1 \\ I_4 & 1 & 1 & 0 & 0 & 0 & 0 \\ & 0 & 1 & 1 & 0 & 1 & 1 \\ & 0 & 0 & 1 & 1 & 0 & 1 \end{array} \right]$$

Extension Columns	**Name**
[0101]	K_5
[0111] [1101] [1111]	D_2
[1011] [1110]	D_3

Table 3a: Single-element extensions of $M(K_5 \backslash e)$

The isomorphisms from these representations of D_2 and D_3 to the previous representations are, respectively,

$$\{1,2,3,4,5,6,7,8,9,10\} \to \{3,4,5,8,2,10,9,6,7,1\}$$

and

$$\{1,2,3,4,5,6,7,8,9,10\} \to \{8,7,9,1,3,4,2,5,10,6\}.$$

Using the matrix representation of $M^*(K_5\backslash e)$ in the previous section we see that up to isomorphism the 3-connected binary single-element extensions are as follows:

$$G = \left[\begin{array}{c|ccccc} & 1 & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 1 & 0 \\ I_5 & 0 & 0 & 1 & 1 & 1 \\ & 0 & 1 & 1 & 0 & 0 \\ & 0 & 1 & 0 & 1 & 1 \end{array}\right] \quad (K'_{3,3})^* = \left[\begin{array}{c|ccccc} & 1 & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 1 & 1 \\ I_5 & 0 & 0 & 1 & 1 & 0 \\ & 0 & 1 & 1 & 0 & 0 \\ & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$E_4 = \left[\begin{array}{c|ccccc} & 1 & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 1 & 1 \\ I_5 & 0 & 0 & 1 & 1 & 0 \\ & 0 & 1 & 1 & 0 & 1 \\ & 0 & 1 & 0 & 1 & 0 \end{array}\right] \quad E_6 = \left[\begin{array}{c|ccccc} & 1 & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & 1 & 0 \\ I_5 & 0 & 0 & 1 & 1 & 1 \\ & 0 & 1 & 1 & 0 & 1 \\ & 0 & 1 & 0 & 1 & 1 \end{array}\right] \quad E_7^* = \left[\begin{array}{c|ccccc} & 1 & 0 & 1 & 0 & 1 \\ & 1 & 0 & 0 & 1 & 1 \\ I_5 & 0 & 0 & 1 & 1 & 1 \\ & 0 & 1 & 1 & 0 & 1 \\ & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

Extension Columns	Name
[00101] [00110] [01100] [01110] [10100] [10101]	$(K_5\backslash e)^* + edge$
[01001] [10010] [11011]	$(K'_{3,3})^*$
[01010] [01011] [10001] [10011] [11001] [11010]	E_4
[00111] [01111] [10111] [11100] [11101] [11110]	E_6
[11111]	E_7^*

Table 3b: Single-element extensions of $M(K_5\backslash e)$

The isomorphisms from these representations of E_4, E_6 and E_7^* to the representations of E_4, E_6 and dual of E_7 are, respectively,

$$\{1,2,3,4,5,6,7,8,9,10\} \to \{3,9,2,6,7,8,10,4,5,1\}$$

$$\{1,2,3,4,5,6,7,8,9,10\} \to \{4,10,1,7,2,8,9,3,5,6\}$$

and

$$\{1,2,3,4,5,6,7,8,9,10\} \to \{3,4,1,9,10,8,5,2,6,7\}$$

Thus we conclude that $M^*(K_5\backslash e)$ has five non-isomorphic binary 3-connected single element extensions, the graph G obtained by adding an edge to $(K_5\backslash e)^*$, the cograph $M^*(K_{3,3})$, and three binary non-regular matroids E_4, E_6, and E_7^*.

Since D_2^*, E_4, and E_7^* have an $M^*(K_5\backslash e)$-minor, it follows from Claim 3 that, if M has a P_9 or P_9^*-minor and rank at least 5, then either they are 3-decomposers for M or M has an E_5 or D_2-minor.

Now, Table 2 implies that all the extensions of E_5 have a D_2-minor, which in turn has a $M(K_5\backslash e)$-minor. Further, all single-element extensions, except A, B, and C, have an E_4-minor, which has an $M^*(K_5\backslash e)$-minor. Moreover, since the extensions of E_5 have a D_2-minor and E_5 is self-dual, all the coextensions have a D_2^*-minor (which has an $M^*(K_5\backslash e)$-minor).

Matrix representations for A, B, and C are given below. In Claim 5 we show that the coextensions of A, B, and C have an $M^*(K_5\backslash e)$-minor.

$$A = \left[\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 0 \\ & 1 & 0 & 1 & 1 & 0 & 0 \\ I_5 & 1 & 1 & 0 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad B = \left[\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 0 & 0 \\ I_5 & 1 & 1 & 0 & 1 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 1 \\ & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \quad C = \left[\begin{array}{c|cccccc} & 0 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 1 & 1 & 0 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Claim 5. If M is a coextension of A, B, or C, then M has an $M^*(K_5\backslash e)$-minor.

Proof. Since E_5 is self-dual and every extension has a D_2-minor, it follows that every coextension has a D_2^*-minor, and consequently an $M^*(K_5\backslash e)$-minor. Suppose M is a coextension of A, B, C. Then a partial matrix representation for M is shown in Figure 2.

$$\left[\begin{array}{c|ccccc|c}
 & 0 & 1 & 1 & 1 & 1 & \\
 & 1 & 0 & 1 & 1 & 0 & \\
I_6 & 1 & 1 & 0 & 1 & 1 & \\
 & 1 & 1 & 1 & 1 & 0 & \\
 & 1 & 1 & 0 & 0 & 0 & \\
\hline
 & & & & & & \\
\end{array}\right]$$

Figure 2. Structure of a coextension of A, B, C

There are three types of rows that may be inserted into the last row on the right-hand side of the matrix in Figure 2.

(i) rows that can be added to E_5 to obtain a coextension with no $M^*(K_5\backslash e)$-minor with a 0 or 1 as the last entry;
(ii) the identity rows with a 1 in the last position;
(iii) and the rows "in-series" to the right-hand side of matrices A, B, C with the last entry reversed.

There are no Type I rows. Type II rows are [100001], [010001], [001001], [000101], and [000011]. Type III rows are specific to the matrices A, B, C. For matrix A they are [011111], [101101], [110110], [111101], [110000]. For matrix B they are [011110], [101101], [110111], [111100], and [110000]. For C they are [011110], [101100], [110111], [111101], and 110000]. Thus we see that only ten rows may be added. Table A1 in the Appendix shows that most of these rows result in matroids that are isomorphic to matroids with an $M^*(K_5\backslash e)$-minor. Only two coextensions must be specifically checked for an $M^*(K_5\backslash e)$-minor: $(C, coextn9)$ and $(C, coextn10)$. Observe that, $(C, coextn9)/12\backslash 1 \cong E_4$, and $(C, coextn10)/12\backslash 10 \cong E_4$. Since E_4 has an $M^*(K_5\backslash e)$-minor, we may conclude these matroids have it too.

Claim 6: *If M has a P_9^*-minor, but no D_2, D_2^* E_4 or E_5-minor, then either P_9 is a 3-decomposer for M or $M \cong D_1^*$.*

Proof. Suppose M is an extension of P_9^*. The extensions of P_9^* are the duals of the extensions of P_9. Thus, from Table 1b they are E_1, E_2, E_3, E_4, E_5, E_6, E_6^* and E_7^*. All of these matroids except E_7^* have a P_9-minor since E_1, E_2, E_3, E_4, and E_5 are self-dual and E_6 and E_6^* are both coextensions of P_9. Since E_7^* has an $M^*(K_5\backslash e)$-minor, we may conclude that M cannot have an E_7^*-minor. So P_9 is a decomposer for M.

Suppose M is a coextension of P_9^*. Then since D_1, D_2, D_3 are extensions of P_9, D_1^*, D_2^*, and D_3^* are coextensions of P_9^*. Of these, M cannot be D_2^* by hypothesis and D_3^* since it has a $M(K_5\backslash e)$-minor. Therefore, we may suppose M has a minor isomorphic to D_1^*.

If $M \cong D_1^*$, then we are done. From Table 1a we see that D_1 is formed by adding just one column to P_9 ([1110]), so any extension of D_1 will have a D_2 or D_3-minor. Thus any coextension of D_1^* will have a D_2^* or D_3^*-minor, which have an $M^*(K_5\backslash e)$-minor.

The extensions of D_1^* are the duals of the coextensions of D_1. Observe from Table 4 that all except the second coextension have a P_9^*-minor. Since the second coextension has an E_7-minor, its dual has an E_7^*-minor. Thus we may conclude that $M \cong D_1^*$. Note that $D_1^* \cong (P_\triangle(F_7, F_7)\backslash z)$.

Matroid	**Coextension Rows**	**Name**	Relevant minors
D_1	[000011] [000101] [001010] [001100] [010010] [010100] [011011] [011101] [100010] [100100] [101011] [101101] [110001] [110111] [111000] [111110]		E_1, E_2, E_3, E_4, E_6^*
	[000110]		E_7
	[000111] [001110] [010110] [011001] [100110] [101001] [110011] [111010]		E_3, E_5, E_6^*, E_7
	[001001] [001111]		E_2, E_6^*
	[001011] [001101]		E_2, E_3 E_5
	[010001] [010011] [010101] [010111] [011000] [011010] [011100] [011110] [10001] [100011] [100101] [100111] [101000] [101010] [101100] [101110]		E_2, E_6^*
	[110000] [110100] [111101] [111111]		E_1, E_2
	[110010] [110110] [111001] [111011]		E_2, E_3
D_2	[000011] [000101] [000110] [001111] [100111] [101000]	A	E_5 E_6^* E_7 $K'_{3,3}$
	[011001]	B	E_5, $K'_{3,3}$,R_{10}
	[010111] [110011] [111010]	C	E_3 E_5 E_6^*, E_7
	[000111]	Z	E_7, R_{10}
	[001001] [100100] [101101]		E_4
	[001010] [001100] [100001] [100010] [101011] [101110]		E_4
	[001011] [001101] [100101] [100110] [101001] [101100]		E_4
	[001110] [100011] [101010]		E_4
	[010001] [011000] [011011] [011101] [110110] [111001]		E_4
	[010010] [010100] [110000] [110101] [111100] [111111]		E_4
	[010011] [010101] [110010] [110111] [111000] [111011]		E_4
	[010110] [011010] [011100] [011111] [110001] [111110]		E_4

Table 4: Single-element coextensions of D_1 and D_2

Claim 7. If M is a coextension of D_2, then either M has an $M^*(K_5\backslash e)$-minor or M is isomorphic to A, B, C, or Z.

Proof. Table 4 verifies that all the single-element coextensions of D_2 except for A, B, C, and Z have an E_4-minor. Further, observe that the choice of rows for Z is just one, so its coextensions will have a minor isomorphic to one of the other matroids. Claim 5 already established that coextensions of A, B, and C have an $M^*(K_5\backslash e)$-minor. So Z does not give rise to new coextensions.

Claims 5, 6, and 7 and the fact that E_7^* has an $M^*(K_5\backslash e)$-minor imply that there are only three possibilities for M: P_9 is a 3-decomposer for M; $M \cong D_1^*$; or M has a minor isomorphic to E_5 or D_2.

Returning to the proof of the theorem, we must show that if M has an E_5 or D_2-minor and no $M^*(K_5\backslash e)$-minor, then the rank of M is bounded above by 5. To do this, let us begin by computing the single-element extensions of A, B, C and Z with no $M^*(K_5\backslash e)$-minor. From Table 2, we may conclude that the only columns that can be added to E_5 to obtain a matroid with no $M^*(K_5\backslash e)$-minor are [00101], [00110], [01011], [01100] [10011], [11001], [11101]. Adding these columns gives us four non-isomorphic single-element extensions of A, B, and C. They are D, E, F, and G shown below (all the extensions of E_5 are shown in Table 5).

$$D = \left[\begin{array}{c|ccccccc} & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ I_5 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \quad E = \left[\begin{array}{c|ccccccc} & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$F = \left[\begin{array}{c|ccccccc} & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \quad G = \left[\begin{array}{c|ccccccc} & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ I_5 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

Suppose M is a coextension of D, E, F, or G. Then the structure of M is shown in Figure 3. Recall from the proof of Claim 5, that there are no Type I rows to add. Adding a Type II or III row (with the exception of [0000011]) causes $M\backslash 12$ to be 3-connected (and there are no such matroids). So the only coextension we must check is the one formed by adding row [0000011]. That

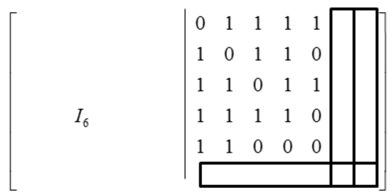

FIGURE 3. Structure of a coextension of D, E, F, G

is the coextensions in which $\{6, 11, 12\}$ is a triad. Let D', E', F', and G' be the coextension of D, E, F, and G, respectively, obtained by coextending by row [0000011]. Then in each case we can find an E_4 minor. In particular, $D'/1\backslash\{3, 11\} \cong E_4$, $E'/1\backslash\{7, 11\} \cong E_4$, $F'/1\backslash\{7, 11\} \cong E_4$, and $G'/1\backslash\{7, 11\} \cong E_4$. Finally, observe that if M is an extension of E_5 of size $k \geq 13$, then for some $e \in \{11, \ldots, k\}$, $M\backslash e$ is 3-connected.

Next, suppose M is a coextension of Z. Observe from Table 5 that Z is an extension of E_7 and R_{10}. In the representation of R_{17} given in the introduction, R_{10} is isomorphic to the first ten columns and Z is isomorphic to the first eleven columns. Let us take that as a representation of Z.

$$Z = \left[\begin{array}{c|cccccc} & 1 & 0 & 0 & 1 & 1 & 0 \\ & 1 & 1 & 0 & 0 & 1 & 1 \\ I_5 & 1 & 1 & 1 & 0 & 0 & 0 \\ & 0 & 1 & 1 & 1 & 0 & 1 \\ & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

R_{10} has two non-isomorphic binary 3-connected single-element extensions, Z and B. Z is obtained by adding any one of the columns [01011], [01101] [10101] [10110] [11010] or [11111] and A is obtained by adding any one of the remaining columns (see [8] for details). Observe from the representation of R_{17} in the Introduction that adding all of the above six columns to Z gives us $R_{17}\backslash e$. Adding one additional column gives us R_{17}.

Let M be a coextension of Z. Observe that R_{10} has two single-element coextensions, A^* and Z^* both of which have a D_2^*-minor (and consequently an $M^*(K_5\backslash e)$-minor. Thus every coextension of R_{10} has an $M^*(K_5\backslash e)$-minor). As before, there are three types of rows that may be added to M. There are no Type I rows. Type II rows are [100001], [010001], [001001], [000101] and [000011]and Type III rows are [100111], [110010], [111001], [011100] and [001110]. Observe that, adding any of the above ten rows to Z gives an isomorphic matroid (Appendix Table A1). Without loss of generality let M be obtained from Z by adding row [000011]. Then, $M/1\backslash 7 \cong E_4$. Therefore, every coextension of Z has an $M^*(K_5\backslash e)$-minor.

It is easy to check that Z has three non-isomorphic single-element extensions, *namely*, D and F mentioned above, and Y shown below (Table 5).

$$Y = \left[\begin{array}{c|ccccccc} & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ I_5 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

Let Y' be coextension of Y formed by adding row [0000011]. Then, $Y'/1\backslash\{2, 7\} \cong E_4$ Thus, we may conclude that if M has an R_{10}-minor and no minor isomorphic to $M^*(K_5\backslash e)$-minor, then M has rank at most 5.

Next, suppose M has a D_2-minor and no minor isomorphic to $M^*(K_5\backslash e)$-minor. Then, by Claim 7 the only coextensions of D_2 with no $M^*(K_5\backslash e)$-minor are A, B, C, and Z. From Table 1a we see that D_2 has two single-element extensions X_1 and X_3 shown below:

$$X_1 = \left[\begin{array}{c|ccccccc} & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ I_4 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 1 & 0 \end{array}\right] X_3 = \left[\begin{array}{c|ccccccc} & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ I_4 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ & 1 & 1 & 1 & 1 & 0 & 1 & 1 \end{array}\right]$$

There are three types of rows that may be added to X_1 and X_3.

(i) the rows that can be added to D_2 to obtain a coextension with no $M^*(K_5\backslash e)$-minor with a 0 or 1 in the last entry. (These are the rows corresponding to A, B, C, Z in Table 4.)
(ii) the identity rows with a 1 in the last position;
(iii) and the rows "in-series" to the right-hand side of matrices X_1 and X_3 with the last entry reversed.

If M is the coextension obtained by adding the first type of row, then $M\backslash 12$ is isomorphic to A, B, C, or Z. Thus M is either D, E, F, G or Y. Type II rows are [1000001], [0100001], [0010001], [0001001], [0000101], [0000011]. Type III rows for X_1 are [011111], [101101], [110110], [111101], [110000] and for X_3 are [011110], [101101], [110111], [111100], and [110000]. For C they are [011110], [101100], [110111], [111101], and 110000]. In each case we were able to find an $M^*(K_5\backslash e)$-minor (Appendix, Table A2).

Lastly, from Table 1a we see that X_1 and X_3 have two non-isomorphic single-element extensions Y_1 and Y_2. Suppose M is a coextension of Y_1 or Y_2. Then M has rank 5 and 13 elements. If we add Type I rows, then $M\backslash 13$ is 3-connected, and if we add Type II or III rows, then $M\backslash 12$ is 3-connected, except when the row added is [00000011]. So only two matroids must be specifically checked for an $M^*(K_5\backslash e)$-minor. They are Y_1 with row [00000011] and Y_2 with row [00000011]. In both cases case the resulting matroid has an $M^*(K_5\backslash e)$-minor. Thus we may conclude that if M has an E_5 or D_2-minor, then the rank of M is at most 5. □

3. Proof of Theorem 1.4

In this section we prove Theorem 1.4. We also give characterizations of the class of binary matroids with no prism-dual minor and the class of binary matroids with no prism and prims-dual minor.

Proof of Theorem 1.4. Theorem 1.3 establishes that the exceptional matroids in the class (i.e. matroids without an exact 3-seperation induced by P_9) are either the infinite families or $(P_\triangle(F_7, F_7)\backslash z)$ or have rank at most 5. Clearly F_7 and F_7^* have no prism minor. Since $M^*(K_5\backslash e)$ has rank 5, all extensions of F_7^* up to $PG(3,2)$ are in the excluded minor class. These are shown in Table 1a.

To complete the proof we must show that R_{17} is the extremal rank 5 binary matroid with no $M^*(K_5\backslash e)$-minor. To do this we will show that if M is a rank-5 binary 3-connected non-regular matroid with no $M^*(K_5\backslash e)$ and an E_5 or D_2-minor, then $M \cong R_{17}$ or its 3-connected restrictions (except P_9^* because it has only 9 elements).

Table 2 implies that the only columns that can be added to E_5 to obtain a matroid with no $M^*(K_5\backslash e)$-minor are those that give A, B, C. That is, columns [00101], [00110], [01011], [01100] [10011], [11001], [11101]. It is straigntforward to check that adding all of these columns gives us a matroid isomorphic to R_{17}. The details are in Table 5.

From Claim 6 of Theorem 1.3, we see that besides A, B, and C, the matroid Z is the only coextension of D_2 with no $M^*(K_5\backslash e)$-minor. As noted earlier, Z is an extension of R_{10} and is obtained by adding any one of the columns [01011], [01101] [10101] [10110] [11010] or [11111] to R_{10}. The only other extension of R_{10} is A. Adding all of the above six columns to Z gives us $R_{17}\backslash e$. Adding one additional column (corresponding to extension A) gives us R_{17}.

One final matter must be checked. It may be possible for R_{17} or one of its deletion-minors to be an extension of the graph $(K_5\backslash e) + edge$ or the cograph $(K'_{3,3})^*$. We must rule out this possibility. To do so, first observe from Table 3a that $M^*(K_5\backslash e)$ has two non-regular extensions $(K_5\backslash e)^* + edge$ and $(K'_{3,3})^*$. Second, observe that E_5 has no minor isomorphic to the prism graph or its dual. Third, Table 2 lists all the 3-connected deletion-minors of A, B, C, making it clear that they have no $M^*(K_5\backslash e)$-minor. Lastly, Table 5 gives the single-element extensions of A, B and C using columns [00101], [00110], [01011], [01100] [10011] [11001] and [11101] (the other columns give an E_4 minor, which has a prism minor). These columns give four 12-element extensions, D, E, F, and G. These 12-element matroids have no graphic nor cographic single-element deletions. So, they belong to the excluded minor class. Moreover, all their extensions will likewise have no graphic nor cographic single-element deletions. We are justified in addding all these columns to E_5 to get R_{17}. Hence proved. □

Matroid	Extension column	Name
A	[**00110**] [01100] [10011]	D
	[**01011**]	E
	[**11001**]	F
	[**11101**]	G
B	[00101] [00110] [01011] [01101]	D
	[11001] [11101]	E
C	[00101] [01011] [10011] [11101]	F
	[00110] [01100]	G
D	[**01011**] [01100] [10011]	H
	[**11001**] [11101]	E
E	[00110] [01100] [10011]	H
	[**11001**]	J
	[**11101**]	K
F	[00110] [01100] [10011] [11101]	I
	[01011]	J
G	[00110] [01100] [10011] [11001]	I
	[01011]	K
H	[**01100**] [10011]	L
	[**11001**] [11101]	M
I	[01011] [01100] [10011] [11101]	M
J	[00110] [01100] [10011] [11101]	M
K	[00110] [01100] [10011] [11001]	M
L	[**10011**]	O
	[**11001**] [11101]	P
M	[01100] [10011] [11101]	P
O	[**11001**] [11101]	Q
P	[10011] [11101]	Q
Q	[**11101**]	R

Table 5: All extensions of E_5 up to R_{17}

Using Table 5, we can identify the internally 4-connected restrictions of R_{17} as all, except B, G and K. Among restrictions of $PG(3, 2)$ all except $K_5\backslash e$, S_8, $AG(3, 2)$, P_9, Z_4, D_1 and X_2 are internally 4-connected. The next corollary follows immediately.

Corollary 3.1. *M is an internally 4-connected binary matroid with no $M^*(K_5\backslash e)$-minor if and only if M is isomorphic to F_7 or an internally 4-connected restriction of $PG(3, 2)$ or R_{17}.* □

The above corollary is the main theorem in [5] by Mayhew and Royle. The matroid they call $AG(3,2) \times U_{1,1}$ is R_{17}. The five matroids they refer to are B, G, K, D_1, X_2.

Theorem 3.2. *Let M be a binary matroid with no prism-minor.*
 (i) *If M is internally 4-connected, then M has rank at most 5, and is isomorphic to a minor of R_{17}*
 (i) *If M is 3-connected but not internally 4-connected, and M has an internally 4-connected minor with at least 6 elements that is not isomorphic to $M(K_4)$, F_7, F_7^* or $M(K_{3,3})$, then M is isomorphic to one of five matroids.*

Using Theorem 1.4 we can also identify members in the dual class (the class of binary matroids with no $M(K_5\backslash e)$-minor). However, we can only conclude that either P_9 or P_9^* are 3-decomposers, instead of the stronger statment that "P_9 is a 3-decomposer." This is because E_7^* has no $M(K_5\backslash e)$-minor nor P_9-minor, but it does admit the 3-separation of its minor P_9^*.

Theorem 3.3. *Suppose M is a 3-connected binary matroid with no $M(K_5\backslash e)$-minor. Then either P_9 or P_9^* is a 3-decomposer or M is isomorphic to one of the following matroids:*
 (i) *$M^*(K_5\backslash e)$, $M(K_{3,3})$, $M^*(K_{3,3})$, or $M(W_r)$ for some $r \geq 3$;*
 (ii) *Z_r, Z_r^*, $Z_r\backslash b_r$, or $Z_r\backslash c_r$, for some $r \geq 4$; or*
 (iii) *F_7, F_7^*, P_9, P_9^**
 (iii) *R_{17}^* or one of its contraction-minors.* □

Proof. Suppose M is a 3-connected binary non-regular matroid with no $M(K_5\backslash e)$-minor. If M is regular, then M is isomorphic to $M^*(K_5\backslash e)$, $M(K_{3,3})$, $M^*(K_{3,3})$, $M(W_r)$ for some $r \geq 3$, or R_{10}. Therefore, suppose M is non-regular. If M has no P_9 or P_9^*-minor, then by Theorem 2.2 (Claim 1), M is isomorphic to F_7, F_7^*, Z_r, Z_r^*, $Z_r\backslash b_r$, or $Z_r\backslash c_r$, for some $r \geq 4$. Thus we may assume M has a P_9 or P_9^*-minor. Note that among the extensions of P_9, D_2 has an $M(K_5\backslash e)$-minor and P_9 is a 3-decomposer for rank-4 binary 3-connected matroids without a D_2-minor. So no further rank 4 matroids are int he class. It follows from Theorem 1.3(iv) that M is isomorphic to R_{17}^* or its contraction minors. □

The next result follows from Theorems 1.3 and 3.3 and the fact that all the rank 5 exceptions and their duals have a minor isomorphic to $M(K_5\backslash e)$ or $M^*(K_5\backslash e)$ except P_9, P_9^*, and E_5.

Theorem 3.4. *Suppose M is a 3-connected binary matroid with no $M(K_5\backslash e)$ or $M^*(K_5\backslash e)$-minor, then either P_9 is a 3-decomposer for M or M is isomorphic to $M(W_r)$ for some $r \geq 3$, $M(K_{3,3})$, $M^*(K_{3,3})$, Z_r, Z_r^*, $Z_r\backslash b_r$, $Z_r\backslash c_r$, for some $r \geq 4$, F_7, F_7^*, P_9, $(P_\triangle(F_7, F_7)\backslash z)$, $(P_\triangle(F_7, F_7)\backslash z)^*$, P_9^*, R_{10}, or E_5.* □

Acknowledgement: A year and half ago, the first author submitted a research proposal outlining matroid representability problems. The unknown reviewer indicated we should tackle this problem. The authors thank the reviewer for highlighting it.

References

(1) G. A. Dirac, Some results concerning the structure of graphs, *Canad. Math. Bull.* **6** (1963) 183-210.
(2) S. R. Kingan, Binary matroids without prisms, prism duals, and cubes, *Discrete Mathematics*, **152** (1996), 211-224.
(3) S. R. Kingan, M. Lemos, Strong Splitter Theorem (submitted).

(4) D. Mayhew, G. Royle, and G. Whittle, The internally 4-connected binary matroids with no $M(K_{3,3})$-minor. *Memoirs of the American Mathematical Society* **981**, American Mathematical Society, Providence, Rhode Island.
(5) D. Mayhew, G. Royle, The internally 4-connected binary matroids with no $M(K_5\backslash e)$-minor (to appear) *Siam Journal on Discrete Mathematics.*
(6) J. G. Oxley, The binary matroids with no 4-wheel minor, *Trans. Amer. Math. Soc.* **154** (1987), 63-75.
(7) J. G. Oxley, *Matroid Theory*, (1992), Oxford University Press, New York.
(8) N. Robertson and P. D. Seymour, Generalizing Kuratowski's Theorem, *Congressus Numerantium* **45** (1984) 129-138.
(9) P. D. Seymour, Decomposition of regular matroids, *J. Combin. Theory Ser. B* **28** , (1980) 305-359.

4. Appendix

Table A1 lists the single-element coextensions of A, B, C, and Z. Table A2 lists the single-element coextensions of X_1 and X_3. Type II and III are marked in red.

Matroid	Name	**Coextension Row**
A	coext 1	[000011] [000101] [001010] [011010] [101111] [111001]
	coext 2	[000110] [110011] [110101]
	coext 3	[000111] [101011] [111011]
	coext 4	[001001][010110] [011111]
	coext 5	[001011] [011011] [100111]
	coext 6	[001100] [011100] [110000]
	coext 7	[001101] [010010] [010100] [011101] [101110] [111000]
	coext 8	[001110] [011000] [101101] [110010] [110100] [111101]
	coext 9	[001111] [011001] [100011] [100101] [101010] [111010]
	coext 10	[010001] [100010] [100100]
	coext 11	[010011] [010101] [100110]
	coext 12	[010111]
	coext 13	[100001] [101000] [111110]
	coext 14	[101001] [110110] [111111]
B	coext 1	[000011] [000101] [000110] [001001] [001010] [001111] [010010] [010100] [010111] [011000] [011011] [011110]
	coext 2	[000111] [001011] [010110] [011010]
	coext 3	[001100] [010001] [011101]
	coext 4	[001101] [001110] [010011] [010101] [011001] [011100]
	coext 5	[100001] [100010] [100100] [101000] [101101] [101110] [110000] [110011] [110101] [111001] [111100] [111111]
	coext 6	[100011] [100101] [101010] [101111] [111000] [111011]
	coext 7	[100110] [101001] [110010] [110100] [110111] [111110]
	coext 8	[100111] [101011] [111010]
C	coext 1	[000011] [000101] [001001] [001111] [010010] [010100] [011000] [011110] [100010] [100100] [101000] [101110] [110011] [110101] [111001] [111111]
	coext 2	[000110] [010111]
	coext 3	[000111] [010110] [100110] [110111]
	coext 4	[001010] [011011]
	coext 5	[001011] [011010] [101010] [111011]
	coext 6	[001100] [011101]
	coext 7	[001101] [011100] [101100] [111101]
	coext 8	[001110] [010011] [010101] [011001]
	coext 9	[010001]
	coext 10	[100001] [110000]
	coext 11	[100011] [100101] [101111] [111000]
	coext 12	[100111]
	coext 13	[101001] [110010] [110100] [111110]
	coext 14	[101011] [111010]
Z	coext 1	[000011] [000101] [001001] [001110] [010001] [011100] [100001] [100111] [110010] [111001]
	coext 2	[000110] [001011] [001101] [010010] [010101] [011000] [011111] [100010] [100100] [101000] [101110] [110000] [110111] [111011] [111100]
	coext 3	[000111] [001010] [001100] [010011] [010100] [011001] [011110] [100011] [100101] [101001] [101111] [110001] [110110] [111010] [111101]
	coext 4	[010110] [010111] [011010] [011011] [101010] [101011] [101100] [101101] [110100] [110101] [111110] [111111]

Table A1: Single-element coextensions of A, B and C

Matroid	Name	Coextension Row
X_1	coext 1	[0000011] [0000101] [0000110] [0001001] [0001010] [0001100] [0010011] [0011100] 0100110] [0101001] [0110101] [0111010]
	coext 2	[000111] [001011] [0001101] [0001110] [1001111] [1010011] [1100110] [1110101]
	coext 3	[0001111]
	coext 4	[0010001] [0010010] [0010100] [0011000] [0100001] [0100010] [0100100] [0101000] [0110111] [0111011] [0111101] [0111110] [1000101] [1001001] [1001100] [1010000] [1010110] [1011010] [1100000] [1100011] [1101010] [1111001] [1111100] [1111111]
	coext 5	[0010101] [0010110] [0011001] [0011010] [0100011] [0100101] [0101010] [0101100] [0110011] [0110110] [01111001] [0111100]
	coext 6	[0010111] [0011011] [0011101] [0011110] [0100111] [0101011] [0101101] [0101110] [0110001] [0110010] [0110100] [0111000] [1000011] [1000110] [1001010] [1010101] [1011001] [1011001] [1011111] [1100101] [1101100] [1101111] [1110000] [1110011] [1110110]
	coext 7	[0011111] [0101111] [0110000]
	coext 8	[1000001] [1000100] [1001000] [10101000] 1011000] [1011110] [110001] [1101000] 1101011] [1111000] [1111011] [1111110]
	coext 9	[100010] [1011101] [1101101] [1110010]
	coext 10	[1000111] [1001011] [1001110] [1010001] [1010111] [1011011] [1100100] [1100111] [1101110] [1110001] [1110100] [1110111]
	coext 11	[1001101] [1010010] [1100010] [1111101]
X_3	coext 1	[0000011] [0000110] [0010001] [0100111] [0101110] [0110000] [0110101] [1010111] [1011011] [1101101] [1110011]
	coext 2	[0000101] [0001001] [0001100] [0011110] [0100001] [0101000] [0111010] [0111111] [1010100] [1011000] [1100010] [1111100]
	coext 3	[0000111] [0001011] [0001110] [0101111] [0110001] [1010011]
	coext 4	[0001101] [0011111] [1100110] [1110100]
	coext 5	[0001111]
	coext 6	[0010010] [1101011] [1111001]
	coext 7	[0010011] [1000111] [1001011] [1001110] [1101111] [1110001]
	coext 8	[0010100] [0011000] [0100010] [0111100] [1000101] [1001001] [1001100] [1011110] [1100001] [1101000] [1111010] [1111111]
	coext 9	[0010101] [0011001] [0100011] [0101010] [0111000] [0111101] [1010110] [1011010] [1100101] [1101100] [1110010] [1110111]
	coext 10	[0010110] [0011010] [0100101] [0101100] [0110010] [0110111] [1010101] [1011001] [1100011] [1101010] [1111000] [1111101]
	coext 11	[0010111] [0011011] [0110011] [1000011] [1000110] [1001010] 1010001] 1100111] [1101110 [1110000] [1110101]
	coext 12	[0011100] [1000001] [1000100] [1001000] 1100000] [1111110]
	coext 13	[0011101] [1100100] [1110110]
	coext 14	[0100100] [0110110] [1011101]
	coext 15	[0100110] [0110100] [1001101] [1011111]
	coext 16	[0101001] [0111011] [1000010] [1010000]
	coext 17	[0101011] [0111001] [1010010]
	coext 18	[1001111]

Table A2: Single-element coextensions of X_1 and X_3